

On similarity and pseudo-similarity solutions of Falkner-Skan boundary layers

M. Guedda and Z. Hammouch*

LAMFA, CNRS UMR 6140, Université de Picardie Jules Verne,
Faculté de Mathématiques et d'Informatique, 33, rue Saint-Leu 80039 Amiens, France

Abstract

The present work deals with the two-dimensional incompressible, laminar, steady-state boundary layer equations. First, we determine a family of velocity distributions outside the boundary layer such that these problems may have similarity solutions. Then, we examine in detail new exact solutions, called *Pseudo-similarity*, where the external velocity varies inversely-linear with the distance along the surface ($u_e(x) = u_\infty x^{-1}$). The analysis shows that solutions exist only for a lateral suction. For specified conditions, we establish the existence of an infinite number of solutions, including monotonic solutions and solutions which oscillate an infinite number of times and tend to a certain limit. The properties of solutions depend on the suction parameter. Furthermore, making use of the fourth-order Runge-Kutta scheme together with the shooting method, numerical solutions are obtained.

Keywords: Similarity solutions, Pseudo-similarity solutions, Falkner-Skan problem, Boundary Layer, Stretching surfaces.

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1. Introduction

In this paper we are concerned with the classical two-dimensional laminar incompressible boundary layer flow past a wedge or a flat plate [28]. For the first approximation, the model is described by the Prandtl equations or the boundary layer equations

$$(1.1) \quad u\partial_x u + v\partial_y u = u_e\partial_x u_e + \nu\partial_{yy}^2 u, \quad \partial_x u + \partial_y v = 0,$$

where (x, y) denote the usual orthogonal Cartesian coordinates parallel and normal to the boundary $y = 0$ (the wall), u and v are the corresponding velocity components, and the constant $\nu > 0$ is the kinematic-viscosity. The function u_e is a given exterior streaming velocity flow which is assumed throughout the paper to be nonnegative function of the single variable x ; $u_e = u_e(x)$, and is such that $u(x, y)$ tends to $u_e(x)$ as $y \rightarrow \infty$. Equations (1.1) can be written in the form

$$(1.2) \quad \partial_y \psi \partial_{xy}^2 \psi - \partial_x \psi \partial_{yy}^2 \psi = u_e \partial_x u_e + \nu \partial_{yyy}^3 \psi,$$

where ψ is the well-known stream function defined by $u = \partial_y \psi, v = -\partial_x \psi$.

*Corresponding author. E-mail: zakia.hammouch@u-picardie.fr

This problem with appropriate external velocity flow has been the main focus of studies of particular exact solutions. Research on this subject has a long history, which dates to the pioneering works by Blasius [6] and Falkner and Skan [13] in which the external velocity is given by

$$(1.3) \quad u_e(x) = u_\infty x^m \quad (u_\infty > 0).$$

Their investigations lead to solutions to (1.2) in the form

$$(1.4) \quad \psi(x, y) = ax^\alpha f(byx^{-\beta}), \quad a, b > 0,$$

where

$$(1.5) \quad \alpha = \frac{m+1}{2} \text{ and } \beta = \frac{1-m}{2}.$$

The equation for f is the well-known equation obtained by Falkner-Skan [13]

$$(1.6) \quad f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) = 0 \quad \text{on } (0, \infty),$$

or, if $m > -1$,

$$(1.7) \quad f''' + f f'' + \sigma(1 - f'^2) = 0 \quad \text{on } (0, \infty),$$

where

$$\sigma = \frac{2m}{m+1}.$$

Such equations occur at wedge flows [28, p. 170]. These equations have received considerable attention in the literature. We refer the reader to the works of Rosenhead [27], Schlichting and Gersten [28], Weyl [30] and Coppel [11] and the references therein. Note that from (1.4) it is easily verified that,

$$\psi(x, y) = \left(\frac{x}{x_0}\right)^\alpha \psi\left(x_0, y \left(\frac{x}{x_0}\right)^{-\beta}\right).$$

This means that a solution $\psi(x, y)$ for y fixed is similar to the solution $\psi(x_0, y)$ at a certain x_0 . This solution is called invariant or similarity solution and the function f is called the shape function or the dimensionless stream function.

The broad goals of this paper is to study equation (1.7) when taking the limit $\sigma \rightarrow -\infty$. This limit case, corresponding to $m = -1$, is that of flow in a two-dimensional divergent channel (or sink flow) [28, p. 170]. We prove that transformation (1.4) is much too restrictive; that is problem (1.1) has no similarity solution and we shall see, by rigorous arguments, that the term $\gamma \log(x)$ must be added to the expression (1.4) and the surface must be permeable with suction to obtain new exact solutions which are not similarity. In addition, we shall prove that there is an infinite number of solutions. These results are given in Section 3. Before this analysis, we shall identify, in Section 2, external flows, such that problem (1.2) may have similarity solutions. The main result of this section indicates that problem (1.2) has solutions under the form (1.4) if the external flow is of the power-law type (1.3).

2. Similarity solutions

In this section we shall obtain external flows such that the partial differential equation (1.2) accompanied by the boundary condition

$$(2.1) \quad \lim_{y \rightarrow \infty} \partial_y \psi(x, y) = \partial_y \psi(x, \infty) = u_e(x),$$

has a solution under the form (1.4), where $\alpha + \beta = 1$.

The main problems, arising in the study of similarity solutions, are related to the existence of the exponents α and β and to the rigorous study of the ordinary differential equation satisfied by the profile f which is, in general, nonlinear. For the layer equation (1.2), the classical approach for identifying α and β is the scaling and transformation group [4]. The essential idea is to seek a and b such that if ψ satisfies (1.2) the new function $\psi_\kappa(x, y) = \kappa^a \psi(\kappa^b x, \kappa y)$ is also a solution.

Let ψ be a stream-function to (1.2) defined by (1.4) where $\alpha + \beta = 1$. Assume first that $\beta \neq 0$. We choose $a = -\frac{\alpha}{\beta}$, $b = \frac{1}{\beta}$, and define $\psi_\kappa(x, y) = \kappa^a \psi(\kappa^b x, \kappa y)$. Hence $a + b = 1$, $\psi \equiv \psi_\kappa$ and

$$L(\psi_\kappa)(x, y) = \kappa^{a+3} L(\psi)(\kappa^b x, \kappa y)$$

for any $\kappa > 0$, where L is the operator defined by

$$L(\psi) = \partial_y \psi \partial_{xy}^2 \psi - \partial_x \psi \partial_{yy}^2 \psi - \nu \partial_{yyy}^3 \psi.$$

According to equation (1.2) we deduce

$$h(x) = \kappa^{a+3} h(\kappa^b x),$$

where $h(x) = u_e(x) \partial_x u_e(x)$. In particular, for fixed $x_0 > 0$

$$h(\kappa^b x_0) = \kappa^{-(a+3)} h(x_0).$$

Setting $x = \kappa^b x_0$ we infer

$$h(x) = x^{-\frac{a+3}{b}} x_0^{\frac{a+3}{b}} h(x_0).$$

Solving the equation

$$(2.2) \quad u_e \frac{du_e}{dx} = x^{-\frac{a+3}{b}} x_0^{\frac{a+3}{b}} h(x_0)$$

yields us

$$(2.3) \quad u_e^2(x) = c_1 x^{2m} + c_2,$$

for all $x > 0$, where $m = \alpha - \beta$ and c_1 and c_2 are constants, for $\beta \neq 0$, since $-\frac{a+3}{b} + 1 = 2(\alpha - \beta)$. For $\beta = 0$, hence $\alpha = m = 1$, the new function

$$\psi_\kappa(x, y) = \kappa^a \psi(\kappa^{-a} x, y),$$

for any fixed $a \neq 0$, is equivalent to ψ and satisfies

$$L(\psi_\kappa)(x, y) = \kappa^a L(\psi)(\kappa^{-a} x, y)$$

for any $\kappa > 0$. Arguing as in the case $\beta \neq 0$ one arrives at (2.3) with $m = 1$. Next, because $\lim_{y \rightarrow \infty} x^{2m} f'(y x^{-\beta})^2 = c_1 x^{2m} + c_2$, the function f'^2 has a finite limit at infinity, which is unique and is given by $c_1 + c_2 x^{-2m}$. This is acceptable only for $c_2 = 0$.

The above result indicates, in particular, that for a prescribed external velocity satisfying (2.3) the real numbers α and β are given by (1.5) and condition (2.3) is necessary and sufficient to obtain similarity solution under the form (1.4) where $\alpha + \beta = 1$. However, for a general external velocity, it is possible to obtain an exact solution which is not similarity solution as it is seen in [24]. In this paper the authors considered

$$u_e(x) = c_1 x^{1/3} + c_2 x^{-1/3}.$$

A stream-function ψ is given by

$$(2.4) \quad \psi(x, y) = x^{2/3} f(yx^{-1/3}) + cx^{-\frac{1}{3}}y, \quad c = \text{const},$$

where f is a solution of

$$(2.5) \quad \nu f''' + \alpha f f'' - (\alpha - \beta) f'^2 = \xi, \quad \xi = \text{const}.$$

3. The pseudo-similarity solutions

In the present section we restrict the attention to the case $m = -1$ and get new solutions to the problem

$$(3.1) \quad \partial_y \psi \partial_{xy}^2 \psi - \partial_x \psi \partial_{yy}^2 \psi = \nu \partial_{yyy}^3 \psi - u_\infty^2 x^{-3},$$

subject to the boundary conditions

$$(3.2) \quad \partial_y \psi(x, 0) = u_w x^{-1}, \quad \partial_x \psi(x, 0) = -v_w x^{-1}, \quad \partial_y \psi(x, \infty) = u_\infty x^{-1},$$

where v_w is a real number ($v_w > 0$ for suction and $v_w < 0$ for injection), u_w and u_∞ are nonnegative and satisfy $u_w < u_\infty$. The velocity distribution $u_e = \frac{u_\infty}{x}$ is found in the case of divergent channel (or sink flow) [28, p. 170]. The analysis of this section is motivated by the work by Magyari, Pop and Keller [26] concerning a boundary-layer flow, over a permeable continuous plane surface, where the x -component velocity tends to zero for y large ($u_\infty = 0$). In [26] the authors showed that if $m = -1$ problem (1.2),(2.1) has no solution in the usual form (1.4). For $u_\infty \neq 0$ and according to Section 2, the function ψ can be written as

$$\psi(x, y) = \sqrt{\nu u_\infty} f \left(\sqrt{\frac{u_\infty}{\nu}} y x^{-1} \right).$$

Since $v_w = \frac{m+1}{2} \sqrt{\nu u_\infty} f(0)$ (see the Appendix) we deduce $v_w = 0$ for $m = -1$ and the following ordinary differential equation for f

$$(3.3) \quad \begin{cases} f''' + f'^2 - 1 = 0, \\ f'(0) = \zeta, f'(\infty) = 1, \end{cases}$$

where $\zeta = \frac{u_w}{u_\infty}$ is in the interval $[0, 1)$ and $f(0)$ can be any real number. Since this problem does not contain f it is convenient to study the equation satisfied by $\theta = f'$; that is

$$(3.4) \quad \begin{cases} \theta'' + \theta^2 - 1 = 0, \\ \theta(0) = \zeta, \theta(\infty) = 1, \end{cases}$$

The stability of equilibrium point $(1,0)$ of (3.4) cannot be determined from the linearization. To analyze the behavior of the nonlinear equation $(3.4)_1$, we observe that

$$E'(t) = 0,$$

where E is the Liapunov function given by

$$E(t) = \frac{1}{2}\theta'(t)^2 + \frac{1}{3}\theta(t)^3 - \theta(t).$$

Then, for some constant c , the following

$$\theta' = \pm\sqrt{2}\left(c + \theta - \frac{1}{3}\theta^3\right)^{1/2},$$

holds. The analysis of the algebraic equation of the phase path in the phase plane reveals that the equilibrium point $(1,0)$ is a center. Hence, Problem (3.4) has no solution for any $\zeta > -1$ except the trivial one $\theta = 1$ (see Fig. 3.1).

Note that if we impose the condition $\theta(\infty) = -1$ instead of $\theta(\infty) = 1$, which is also of physical interest, it is easy to see that, for any $\zeta \leq 2$, there exists a unique solution up to translation. This solution satisfies

$$\frac{1}{2}\theta'(t)^2 + \frac{1}{3}\theta(t)^3 - \theta(t) = \frac{2}{3},$$

and we find that

$$\theta(t) = 2 - 3 \tanh^2 \left[\pm t/\sqrt{2} + \operatorname{arctanh} \left\{ (2 - \zeta)/3 \right\}^{1/2} \right].$$

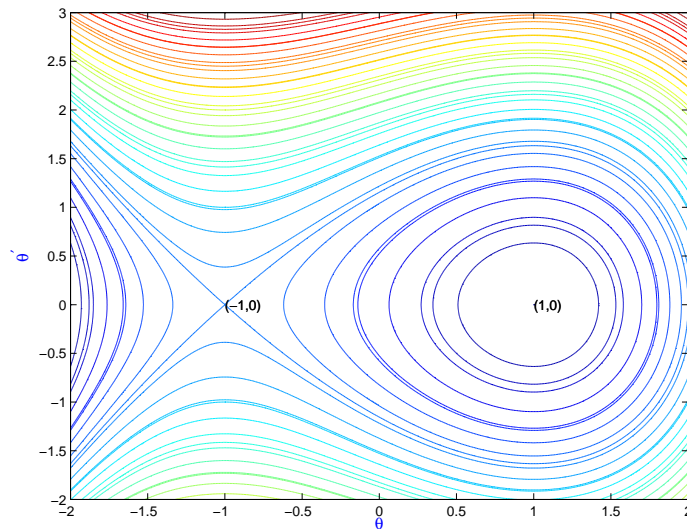


Fig. 3.1 Classification of solutions of $\theta'' + \theta^2 - 1 = 0$ according to $\theta(0)$ and $\theta'(0)$.

To obtain exact solutions to (3.1),(3.2), we look for “*pseudo-similarity*” solutions under the form

$$(3.5) \quad \psi(x, y) = aF(x, byx^{-1}),$$

where $a = \sqrt{\nu u_\infty}$ and $b = \sqrt{\frac{u_\infty}{\nu}}$. Assuming $F(x, t) = f(t) + H(x)$ one sees that (cf. Appendix)

$$(3.6) \quad \begin{cases} f''' + \gamma f'' + f'^2 - 1 = 0, \\ f'(0) = \zeta, \quad f'(\infty) = 1, \end{cases}$$

and

$$H(x) = \gamma \log x + c,$$

where $\gamma = v_w (\nu u_\infty)^{-1/2}$ and c is constant. Without loss of generality we may take $c = 0$, since $\psi(x, y) = \sqrt{\nu u_\infty} f(t) + v_w \log(x) + \sqrt{\nu u_\infty} c$ satisfies (3.1) – (3.2) for any real number c . Note that γ plays the role of suction/injection parameter.

To study (3.6) it is more convenience to consider the second ordinary differential equation

$$(3.7) \quad \begin{cases} \theta'' + \gamma \theta' + \theta^2 - 1 = 0, \\ \theta(0) = \zeta, \quad \theta(\infty) = 1, \end{cases}$$

where $0 \leq \zeta < 1$ and $\gamma \neq 0$. In fact, the real number γ will be taken in $(0, \infty)$. The existence of solutions to (3.7) will be proved by means of shooting method. Hence, the boundary condition at infinity is replaced by the condition $\theta'(0) = d$, where d is a real number. For any d the new initial-value problem has a unique local solution θ_d defined in the maximal interval of the existence, say $(0, T_d)$, $T_d \leq \infty$. We shall see that for an appropriate d the solution θ_d is global and satisfies

$$(3.8) \quad \theta_d(\infty) = 1.$$

A simple analysis in the phase plane shows that problem (3.6) may have solutions only for $\gamma > 0$. In fact, the ordinary differential equation in (3.6) is considered as a nonlinear autonomous system in \mathbb{R}^2 , with the unknown (θ, θ') , mainly

$$(3.9) \quad \begin{cases} \theta' = \varphi, \\ \varphi' = -\gamma \varphi + 1 - \theta^2, \end{cases}$$

subject to the boundary condition

$$(3.10) \quad \theta(0) = \zeta, \quad \varphi(0) = d.$$

The linear part of the above system at $(1, 0)$ is the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -2 & -\gamma \end{pmatrix}.$$

The eigenvalues of J are

$$\lambda_1 = \frac{-\gamma - \sqrt{\gamma^2 - 8}}{2}, \quad \lambda_2 = \frac{-\gamma + \sqrt{\gamma^2 - 8}}{2},$$

if $\gamma \geq \sqrt{8}$ and for $|\gamma| < \sqrt{8}$,

$$\lambda_3 = \frac{-\gamma - i\sqrt{8 - \gamma^2}}{2}, \quad \lambda_4 = \frac{-\gamma + i\sqrt{8 - \gamma^2}}{2}.$$

Therefore, the hyperbolic equilibrium point $(1, 0)$ is asymptotically stable if γ is positive and unstable for negative γ . In particular problem (3.5) has no nontrivial solutions if $\gamma < 0$. If $\gamma > 0$ we deduce from the above that there exists $\delta > 0$ such that for any d and ζ satisfying $d^2 + (\zeta - 1)^2 < \delta^2$ the local solution θ_d is global and satisfies (3.8). In the following we construct solution to (3.6) where the condition $d^2 + (\zeta - 1)^2 < \delta^2$ is not necessarily required. For a mathematical consideration the parameter ζ will be taken in $(-1, \sqrt{3}]$. The following result deals with nonnegative values of ζ . Let us consider a real number d such that

$$(3.11) \quad d^2 \leq 2\zeta\left(1 - \frac{\zeta^2}{3}\right),$$

where $0 \leq \zeta \leq \sqrt{3}$. We shall see that any local solution of (3.9), (3.10) is global and satisfies (3.8). To this end we consider again the Liapunov function $E(\theta(t), \varphi(t)) = \frac{1}{2}\varphi(t)^2 + \frac{1}{3}\theta^3 - \theta$. Along an orbit we have

$$\frac{d}{dt}E(\theta(t), \varphi(t)) = -\gamma\varphi(t)^2 \leq 0.$$

Hence

$$E(\theta_d(t), \theta'_d(t)) \leq E(\zeta, d),$$

for any $t < T_d$. On the other hand, from (3.9) and (3.10), there exists $t_0 > 0$, small, such that θ_d is positive on $(0, t_0)$. Assume that θ_d vanishes at some $t_1 > t_0$ and suppose that $\theta'_d(t_1) \neq 0$. Because

$$E(\zeta, d) \geq E(\theta_d(t), \theta'_d(t)) \geq \frac{1}{2}\theta'_d(t_1)^2,$$

for all $0 \leq t \leq t_1$. we deduce $\frac{1}{2}d^2 > \zeta(1 - \frac{1}{3}\zeta^2)$, which contradicts (3.11). Therefore $\theta'_d(t_1) = 0$. In this case we deduce from the equation of θ_d that $\theta''_d(t_1) = 1$ and then θ_d is nonnegative on a some neighbourhood of t_1 . Consequently the local solution is nonnegative as long as there exists. To show that θ_d is global we note that

$$E(\zeta, d) \geq \frac{1}{2}\theta'_d(t)^2 + \frac{1}{3}\theta_d^3(t) - \theta_d(t) \geq -\frac{2}{3},$$

for all $t \leq T_d$, since θ_d is nonnegative. Hence θ_d and (then) θ'_d are bounded. Consequently θ_d is global. It remains to show that θ goes to unity at infinity. To this end we use the *Bendixon Criterion*. Let \mathcal{T} be the trajectory of (θ_d, θ'_d) in the phase plane $(0, \infty) \times \mathbb{R}$ for $t \geq 0$ and let $\Gamma^+(\mathcal{T})$ be its w -limit set at infinity. From the boundedness of \mathcal{T} it follows that $\Gamma^+(\mathcal{T})$ is a nonempty connected and compact subset of $(0, \infty) \times \mathbb{R}$ (see, for example [1, p 226]). Moreover $(-1, 0) \notin \Gamma^+(\mathcal{T})$, since θ_d is nonnegative. Note that if $\Gamma^+(\mathcal{T})$ contains the equilibrium point $(1, 0)$ then $\Gamma^+(\mathcal{T}) = \{(1, 0)\}$, since $(1, 0)$ is asymptotically stable. Assume that $(1, 0) \notin \Gamma^+(\mathcal{T})$, Applying the *Poincaré–Bendixon Theorem* [14, p 44] we deduce that $\Gamma^+(\mathcal{T})$ is a cycle, surrounding $(1, 0)$. To finish, we shall prove the nonexistence of such a cycle. We define $P(\theta, \varphi) = \varphi$, $Q(\theta, \varphi) = -\gamma\varphi + 1 - \theta^2$, $\varphi = \theta'_d$ and $\theta = \theta_d$. The function (θ, φ) satisfies the system $\theta' = P(\theta, \varphi)$, $\varphi' = Q(\theta, \varphi)$. Let D be the bounded domain of the (θ, φ) -plane with boundary Γ^+ . As P and Q are regular we deduce, via the *Green–Riemann Theorem*,

$$(3.12) \quad \int \int_D (\partial_\varphi Q + \partial_\theta P) d\varphi d\theta = \int_{\Gamma^+} (Qd\theta - Pd\varphi) = 0,$$

thanks to the system satisfied by (θ, φ) . But $\partial_\varphi Q + \partial_\theta P = \gamma$ which is positive. We get a contradiction. To complete our analysis, we shall determine a basin of the critical point $(1, 0)$. Let

$$\mathcal{P} = \left\{ (\zeta, d) \in \mathbb{R}^2 : \zeta > -1, \frac{1}{2}d^2 + \zeta \left(\frac{1}{3}\zeta^2 - 1 \right) < \frac{2}{3} \right\}.$$

Let us consider a one-parameter of family of curves defined by

$$E(\theta, \varphi) = \frac{1}{2}\varphi^2 + \frac{1}{3}\theta^3 - \theta = C,$$

where C is a real parameter. Note that, in the phase plane, this family is solution curves of system (3.4). The curve $\varphi^2 = 2\theta - \frac{2}{3}\theta^3 + \frac{4}{3}$, corresponding to $C = \frac{2}{3}$, goes through the point $(2, 0)$ and has the saddle $(-1, 0)$ ($\gamma = 0$) as its α and w -limit sets. We note this solution curve by \mathcal{H} , which is, in fact, an homoclinic orbit and define a separatrix cycle for (3.4). We shall see that the bounded open domain with the boundary \mathcal{H} is an attractor set for $(1, 0)$ of system (3.9) where $\gamma > 0$. This domain is given by $E(\theta, \varphi) = C, \theta > -1$, for all $-\frac{2}{3} \leq C < \frac{2}{3}$, which is \mathcal{P} . As $\frac{d}{dt}E \leq 0$ any solution, with initial data in \mathcal{P} cannot leave \mathcal{P} . By *LaSalle invariance principle* we deduce that for any (ζ, d) in \mathcal{P} the w -limit set, $\Gamma^+(\zeta, d)$ is a nonempty, connected subset of $\mathcal{P} \cap \{\varphi = 0\}$, (see [1, p. 234]). However, if $\theta \neq 1, \varphi = 0$ is a transversal of the phase-flow, so the w -limit set is $\{(1, 0)\}$. This means that \mathcal{P} is a basin of the critical point $(1, 0)$ of (3.9).

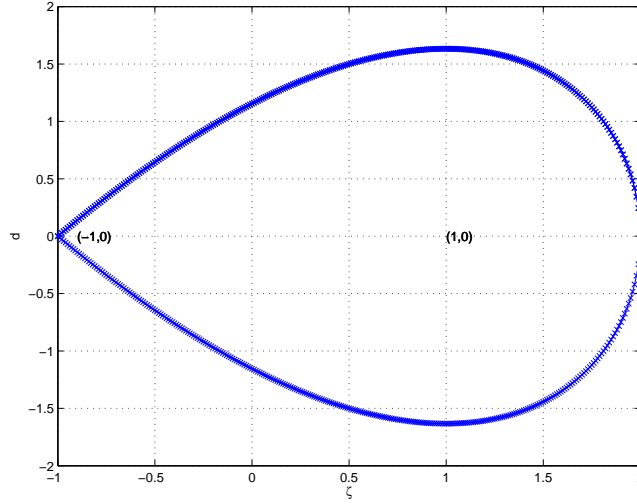


Fig. 3.2 A basin of attraction \mathcal{P} of the critical point $(1, 0)$.

4. Numerical results

In this section numerical solutions of the boundary-value problem (3.7) are obtained by using the fourth-order Runge-Kutta scheme with the shooting method. We plot the dimensionless velocity θ in term of the similarity variable t , for various value of the shooting parameter d .

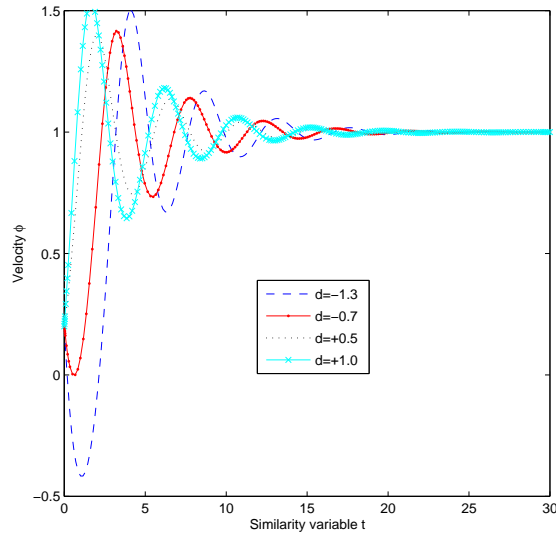


Fig. 4.1 Velocity profiles in terms of $d = \theta'(0)$ for fixed $\zeta = 0.2$ and $\gamma = 0.5$

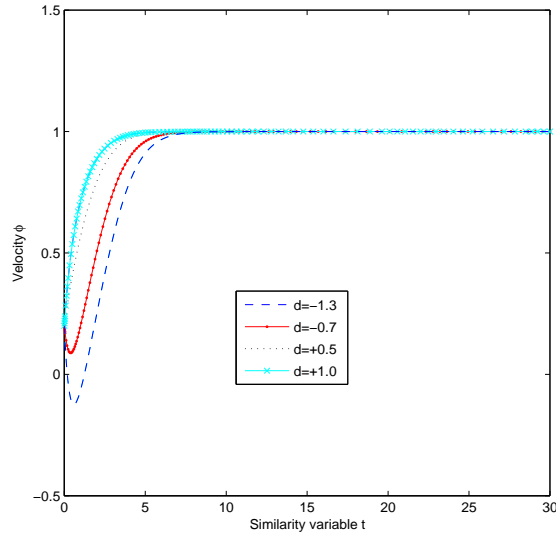


Fig. 4.2 Velocity profiles in terms of $d = \theta'(0)$ for fixed $\zeta = 0.2$ and $\gamma = 2.83$

5. Conclusion

In this work the laminar two-dimensional steady incompressible, boundary layer flow past a moving plane is considered. It has been shown that the problem has solutions having a similarity form if the velocity distribution outside the boundary layer is proportional to x^m , for some real number m . In the second part of this paper, we are interested in question of existence of solutions in the case where the external velocity is the inverse-linear function; $m = -1$. This situation occurs in the case of sink flow. To obtain exact solutions the stream function ψ is written under the form

$$(5.1) \quad \psi(x, y) = \sqrt{\nu u_\infty} f(t) + v_w \log(x).$$

It is shown that the ordinary differential equation satisfied by f has multiple solutions for any v_w positive and no solution can exist if $v_w \leq 0$. A sufficient condition for the existence is derived:

$$(5.2) \quad \zeta > -1, \quad \frac{1}{2}f''(0)^2 + \zeta \left(\frac{\zeta^2}{3} - 1 \right) < \frac{2}{3}.$$

We have obtained two family of solutions according to $\gamma = v_w (\mu u_\infty)^{-1/2}$. If $\gamma \geq \sqrt{8}$, f' is monotonic and goes to 1 at infinity, but if $0 < \gamma < \sqrt{8}$, we have a stable spiral. The function f' oscillates an infinite number of times and goes to 1. So if we are interested in solutions to (3.7) such that

$$-1 < f' < 1$$

we must take u_w, v_w and $u_\infty > 0$ satisfying $-u_\infty < u_w < u_\infty$ and $v_w > (8\nu u_\infty)^{1/2}$.

Condition (4.1) indicates also that for the same positive value of the suction parameter the permeable wall stretching with velocity $u_w x^{-1}$, $u_w > 0$ has multiple boundary-layer flows. Every flow is uniquely determined by the dimensionless skin friction $f''(0)$ which can be any real number in the interval $\left(-\sqrt{\frac{4}{3} + 2\zeta(1 - \zeta^2/3)}, \sqrt{\frac{4}{3} + 2\zeta(1 - \zeta^2/3)} \right)$, where $\zeta = u_w u_\infty^{-1}$. The case $u_\infty = 0$ was considered by Magyari, Pop and Keller [26]. The authors showed, by numerical solutions, that the boundary layer flow exists only for a large suction parameter ($\gamma \geq 1.079131$).

The existence of exact solutions of the Falkner-Skan equation under the present condition was discussed by Rosenhead [27, pp. 244–246] who mentioned that these results may be obtained by rigorous arguments which, in fact, motivated the present work. We note, in passing, that it is possible to obtain solutions if the the skin friction satisfies $|f''(0)| > \sqrt{\frac{4}{3} + 2\zeta(1 - \zeta^2/3)}$.

Appendix

Let us now derive problem (3.5). We assume that the external velocity is given by $u_e(x) = u_\infty x^m$, where m is not necessary equal to -1. We recall that the stream function satisfies the following equation

$$(5.3) \quad \partial_y \psi \partial_{xy}^2 \psi - \partial_x \psi \partial_{yy}^2 \psi = \nu \partial_{yyy}^3 \psi + m u_\infty^2 x^{2m-1},$$

with the boundary conditions

$$(5.4) \quad \partial_y \psi(x, 0) = u_w x^m, \quad \partial_x \psi(x, 0) = -v_w x^{\frac{m-1}{2}}, \quad \partial_y \psi(x, \infty) = u_\infty x^m.$$

To obtain exact solutions to (5.3), (5.4), we look for “*pseudo-similarity*” solutions under the form

$$(5.5) \quad \psi(x, y) = a x^\alpha F(x, b y x^{-\beta}).$$

where $\alpha = \frac{m+1}{2}$, $\beta = -\frac{m-1}{2}$, $a = \sqrt{\nu u_\infty}$ and $b = \sqrt{\frac{u_\infty}{\nu}}$. Inserting (4.4) into (4.2), (4.3) leads to

$$(5.6) \quad \begin{cases} F''' + \frac{1+m}{2} F F'' - m(F'^2 - 1) + x(F'' \partial_x F - F' \partial_x F') = 0, \\ F'(x, 0) = \zeta, \quad F'(x, \infty) = 1, \\ \frac{1+m}{2} F(x, 0) + x \partial_x F(x, 0) = \frac{v_w}{\sqrt{\nu u_\infty}}, \end{cases}$$

where the primes denote partial differential with respect to $t = \sqrt{\frac{u_\infty}{\nu}} y x^{\frac{m-1}{2}}$. By writing

$$F(x, t) = f(t) + H(x),$$

we find

$$(5.7) \quad \begin{cases} f''' + \frac{1+m}{2} f f'' - m (f'^2 - 1) + f'' (xH' + \frac{1+m}{2} H) = 0, \\ \frac{1+m}{2} f(0) + \frac{1+m}{2} H(x) + xH'(x) = \frac{v_w}{\sqrt{\nu u_\infty}}, \quad f'(0) = \zeta \in [0, 1), \quad f'(\infty) = 1. \end{cases}$$

Hence, there exists a real number γ such that

$$(5.8) \quad \begin{cases} f''' + \frac{1+m}{2} f f'' - m (f'^2 - 1) + \gamma f'' = 0, & t > 0, \\ xH' + \frac{1+m}{2} H = \gamma, & x > 0, \\ \frac{1+m}{2} f(0) + \gamma = \frac{v_w}{\sqrt{\nu u_\infty}}, & f'(0) = \zeta, \quad f'(\infty) = 1. \end{cases}$$

First, let us assume that $m \neq -1$. The solution H is given by

$$H(x) = cx^{-\frac{1+m}{2}} + \frac{2\gamma}{1+m},$$

where c is a constant, and then $\psi(x, y) = ac + ax^{\frac{1+m}{2}} \left(f(t) + \frac{2\gamma}{1+m} \right)$. The new function $g = f + \frac{2\gamma}{1+m}$ satisfies the Falkner-Skan equation. Thereafter, we will assume that $m = -1$ and this leads to

$$(5.9) \quad \begin{cases} f''' + \gamma f'' + f'^2 - 1 = 0, \\ f'(0) = \zeta, \quad f'(\infty) = 1, \quad \gamma = \frac{v_w}{\sqrt{\nu u_\infty}}, \end{cases}$$

and

$$H(x) = \gamma \log x + c, \quad c = \text{const.}$$

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